Journal of Statistical Physics, Vol. 87, Nos. 1/2, 1997

The Oscillatory Behavior of the High-Temperature Expansion of Dyson's Hierarchical Model: A Renormalization Group Analysis

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Received July 23, 1996; final October 22, 1996

We calculate 800 coefficients of the high-temperature expansion of the magnetic susceptibility of Dyson's hierarchical model with a Landau–Ginzburg measure. Log-periodic corrections to the scaling laws appear as in the case of an Ising measure. The period of oscillation appears to be a universal quantity given in good approximation by the logarithm of the largest eigenvalue of the linearized RG transformation, in agreement with a possibility suggested by Wilson and developed by Niemeijer and van Leeuwen. We estimate γ to be 1.300 (with a systematic error of the order of 0.002), in good agreement with the results obtained with other methods, such as the ε-expansion. We briefly discuss the relationship between the oscillations and the zeros of the partition function near the critical point in the complex temperature plane.

KEY WORDS: Renormalization group; critical exponents; hierarchical models; high-temperature expansion; Ising models; epsilon expansion.

1. INTRODUCTION

A possible way of testing our understanding of second-order phase transitions consists in calculating the critical exponents as accurately as possible. Ideally, one would like to use several independent methods and obtain an agreement within small errors. The renormalization group method[1] has provided several approximate methods to calculate the critical exponents of lattice models in various dimensions. On the other hand, the same exponents can be estimated from the analysis of high-temperature series.[3,4] Showing that these methods give precisely the same answers has been a

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0022-4715/97/0400-0363$12.50/0

challenging problem.[2] In general, one would expect that a well-established discrepancy could either reveal new aspects of the critical behavior of the model considered or point out the inadequacy of some of the methods used.

In order to carry through this program, one needs to overcome technical difficulties which are specific to the methods used. An important problem with the high-temperature expansion[6] is that one needs *much* longer series than the ones available[3] (which do not go beyond order 25 in most of the cases) in order to make precise estimates. On the other hand, a problem specific to the renormalization group method is that the practical implementation of the method usually requires projections into a manageable subset of parameters characterizing the interactions.

It is nevertheless possible to design a nontrivial lattice model,[8] referred to hereafter as Dyson's hierarchical model (in order to avoid confusion with other models also called "hierarchical"), which can be seen as an approximate version of nearest neighbor models and for which these two technical difficulties can be overcome. For Dyson's hierarchical model the renormalization group transformation reduces to a simple integral equation involving only the local measure. This simplicity allows one to control rigorously[9] the renormalization group transformation and to obtain accurate estimates of the eigenvalues of the linearized renormalization group transformation.[10] More recently we have shown that the recursion formula can be put in a form[11, 12] suitable for the calculation of the high-temperature expansion to very large order. Consequently, Dyson's hierarchical model is well suited to compare the ε-expansion and the high-temperature expansion. Note that unlike the ε-expansion, the high-temperature expansion depends on the choice of a local measure of integration for the spin variables (e.g., an Ising or Landau–Ginzburg measure). In order to make this choice explicit when necessary, we will, for instance, speak of Dyson's hierarchical Ising model if we are using an Ising measure.

In a recent publication[12] we reported results concerning the high-temperature expansion of Dyson's hierarchical Ising model. We found clear evidence for oscillations in the quantity used to estimate the critical exponent γ, called the extrapolated slope (see Section 3). When using a log scale for the order in the high-temperature expansion these oscillations become regularly spaced. We provided two possible interpretations. The first is that the eigenvalues of the linearized renormalization group are complex. The

[2] There is a large amount of literature on this subject; for references see, e.g., ref. 5.

[3] B. Nickel, private communication to A. Guttmann, reported in ref. 4, p. 9. For recent calculations see, e.g., ref. 7.

second is that the eigenvalues stay real, but the constants appearing in the conventional parametrization of the magnetic susceptibility should be replaced by functions of $\beta_c - \beta$ invariant under the rescaling of $\beta_c - \beta$ by λ_1, the largest eigenvalue of the linearized renormalization group transformation. Hereafter, we refer to this explanation as "the second possibility." This second possibility was mentioned twice by Wilson[1] and developed systematically by Niemeijer and van Leeuwen.[13]

In ref. 12 we gave several arguments against the first possibility. A more convincing argument is given in Section 7: explicit calculations of the first 14 eigenvalues of the linearized renormalization group transformation not relying on the ε or high-temperature expansion show no evidence for complex eigenvalues of the linearized transformation. In addition, all the results presented below support the second possibility.

In this paper we report the results of calculations of the high-temperature expansion of the magnetic susceptibility of Dyson's hierarchical model up to order 800 with a Landau–Ginzburg measure. These calculations provide good evidence that the oscillations appear with a universal frequency given by the second possibility[1, 13] discussed above, but with a measure-dependent phase and amplitude. Before going into the technical details related to the analysis of the series, we would like to state additional conclusions. First, we found no significant discrepancy between the high-temperature expansion and the ε-expansion. Second, with the existing methods, the high-temperature expansion appears as a rather inefficient way to estimate the critical exponents of Dyson's hierarchical model. Third, the high-temperature expansion reveals small oscillatory corrections to the scaling laws which cannot be detected from the study of the *linearized* renormalization group transformation.

These conclusions were reached after a rather lengthy analysis. The second possibility introduces potentially an infinite number of Fourier coefficients and it is useful to first work with simplified examples in order to develop a strategy to fit the data with as few unknown parameters as possible. Solvable models where the second possibility is realized were proposed in ref. 15. These models are sometimes called "Ising hierarchical lattice models" and should not be confused with Dyson models. Further analysis of these models shows that the zeros of the partition function in the complex temperature plane are distributed on the (very decorative) Julia set[16] of a rational transformation. In particular, it is possible to relate the oscillations to poles of the Mellin transform located away from the real axis at the ferromagnetic critical point. In addition, the calculation of the amplitude of oscillation for these models illustrates a feature which we believe is rather general, namely that the oscillations tend to "hide" themselves: large frequencies imply (exponentially) small amplitudes.

This paper is organized as follows. In Section 2 we specify the models considered and the methods used for the calculations. In Section 3 we explain how to estimate the critical exponent γ using the so-called extrapolated slope.[6] We discuss the effects of the new oscillatory terms on this quantity, using assumptions which are motivated in subsequent sections. In Section 4 we show that despite a large amplification, the systematic and numerical errors on the coefficients play no role in our discussion of the oscillations of the extrapolated slope. This section also provides a test of our calculation method in an explicitly solvable case, namely Dyson's hierarchical Gaussian model.

Inspired by the Ising hierarchical lattice models and the analytical form of the one-loop Feynman diagrams for Dyson's hierarchical model, we designed a simple mathematical function with a singularity corrected by log-periodic oscillations. This function is defined in Section 5. Its power singularity, as well as the frequency, amplitudes, and phases of oscillations, can be explicitly calculated. We then show that these quantities can be extracted in good approximation from a numerical analysis of the extrapolated slope associated with the Taylor expansion of the function about a nonsingular point. In Section 6 we apply the methods developed in Section 5 to fit the extrapolated slope associated with the various high-temperature expansions calculated. The analysis is complicated by the fact that the $1/m$ corrections to the large-m expansion, m being the order in the high-temperature expansion, are enhanced by a factor which is approximately 160. We start with five-parameter fits, which give robust values for the critical exponent γ and the frequency of oscillation ω. From the study of the errors one can design fits with one or two more parameters which have smaller systematic errors and which are reasonably stable under small changes in the fitting interval or in the initial guesses for the values of the parameters.

The results of the numerically stable fits are discussed in Section 7. The linear relation between ω and γ predicted by the second possibility is well obeyed and the value of γ is in good agreement with the value obtained with the ε-expansion, which we have checked using independent methods. All results agree within errors of the order 0.002. We have thus succeeded in finding a theoretical framework in which the new and existing results fit together. Many questions remain: What is the origin of the oscillation? Can we calculate the amplitudes of oscillation directly? Are similar phenomena present for models with nearest neighbor interactions? If the example of the solvable Ising hierarchical lattice models can be used as a guide, these questions require a better understanding of the susceptibility in the complex temperature plane. These questions are briefly discussed in Section 8. In particular, we give preliminary results concerning the zeros of the partition function in the complex temperature plane which suggests an accumulation of zeros near the critical point.

2. RECURSIVE CALCULATION OF THE HIGH-TEMPERATURE EXPANSION

In this section we describe Dyson's hierarchical model and the methods used to calculate the high-temperature expansion of the magnetic susceptibility. The models considered here have 2^n sites. Labeling the sites with n indices $x_n,..., x_1$, each index being 0 or 1, we can write the Hamiltonian as

$$H = -\frac{1}{2} \sum_{l=1}^{n} \left(\frac{c}{4}\right)^{l} \sum_{x_n,..., x_{l+1}} \left(\sum_{x_l,..., x_1} \sigma_{(x_n,..., x_1)} \right)^2 \tag{2.1}$$

The free parameter c which controls the strength of the interactions is set equal to $2^{1-2/D}$ in order to approximate a nearest neighbor model in D dimensions. In this paper we *only* consider the case $D = 3$. The spins $\sigma_{(x_n,..., x_1)}$ are integrated with a local measure which needs to be specified. In the following we consider the Ising measure, where the spins take only the values ± 1, and measures where the spin variables are integrated with a weight $\exp(-A\sigma^2 - B\sigma^4)$, which we call Landau–Ginzburg measures. In the particular case $B = 0$ we obtain a Gaussian measure. In the following we have used $A = 1/2$ with $B = 0.1$ or $B = 1$.

The integrations can be performed iteratively using a recursion formula studied in ref. 9. Our calculation uses the Fourier transform of this recursion formula with a rescaling of the spin variable appropriate to the study of the high-temperature fixed point.[11] It amounts to the repeated use of the recursion formula

$$R_{l+1}(k) = C_{l+1} \exp\left[-\frac{1}{2} \beta \left(\frac{c}{2}\right)^{l+1} \frac{\partial^2}{\partial k^2} \right] \left[R_l\left(\frac{k}{\sqrt{2}}\right) \right]^2 \tag{2.2}$$

which is expanded to the desired order in β.

The initial condition for the Ising measure chosen here is $R_0 = \cos(k)$. For the Landau–Ginzburg measure the coefficients in the k-expansion have been evaluated numerically. The constant C_{l+1} is adjusted in such a way that $R_{l+1}(0) = 1$. After repeating this procedure n times we can extract the finite-volume magnetic susceptibility $\chi_n(\beta) = 1 + b_{1,n}\beta + b_{2,n}\beta^2 + \cdots$ from the Taylor expansion of $R_n(k)$, which reads $1 - (1/2) k^2 \chi_n + \cdots$. This method has been presented for the Ising measure in ref. 11 and checked using results obtained with conventional graphical methods.[17] In the calculations presented below we use $n = 100$, which corresponds to a number of sites larger than 10^{30}. The errors associated with the finite volume are negligible compared to the errors associated with numerical roundoffs, as explained in Section 4.

3. THE EXTRAPOLATED SLOPE

In order to estimate γ, we will use a quantity called the extrapolated slope[5] and denoted $\hat{S}_m$ hereafter. The justification for this will be made clear after we recall its definition. First, we define $r_m = b_m/b_{m-1}$, the ratio of two successive coefficients. We then define the normalized slope S_m and the extrapolated slope $\hat{S}_m$ as

$$\begin{aligned} S_m &= -m(m-1)(r_m - r_{m-1})/[mr_m - (m-1)\, r_{m-1}] \\ \hat{S}_m &= mS_m - (m-1)\, S_{m-1} \end{aligned} \tag{3.1}$$

In the conventional description[14] of the renormalization group flow near a fixed point with only one eigenvalue $\lambda_1 > 1$ the magnetic susceptibility can be expressed as

$$\chi = (\beta_c - \beta)^{-\gamma}\,(A_0 + A_1(\beta_c - \beta)^{\Delta} + \cdots) \tag{3.2}$$

with $\Delta = |\ln(\lambda_2)|/\ln(\lambda_1)$ and λ_2 being the largest of the remaining eigenvalues. It is usually assumed that these eigenvalues are real. When this is the case, one finds[6] that

$$\hat{S}_m = \gamma - 1 + Bm^{-\Delta} + O(m^{-2}) \tag{3.3}$$

Remarkably, the $1/m$ corrections coming from analytic contributions have disappeared, justifying the choice of this quantity. Instead of this monotonic behavior, oscillations with a logarithmically increasing period were observed in ref. 12. Equation (3.3) was then used, allowing B and Δ to be complex and selecting the real part of the modified expression. This introduces two new parameters, and the parametrization of the extrapolated slope becomes

$$\hat{S}_m = \gamma - 1 - a_1 m^{-a_2} \cos(\omega \ln(m) + a_3) \tag{3.4}$$

This parametrization allows one to obtain good-quality fits, provided that m is not too small.

This parametrization is compatible with two interpretations. The first one is that the eigenvalues of the linearized renormalization group are complex. We have given[12] several general arguments against this possibility and an explicit calculation reported in Section 6 makes this possibility quite implausible. The second possibility[1, 13] we have considered is that the eigenvalues stay real, but the constants A_0 and A_1 in Eq. (3.2) are replaced by functions of $\beta_c-\beta$ invariant under the rescaling of $\beta_c-\beta$ by a factor $(\lambda_1)^l$, where l is any positive or negative integer. This invariance implies that these functions are periodic functions in $\log(\beta_c-\beta)$ with period $\log(\lambda_1)$ and can be expanded in integer powers of $(\beta_c-\beta)^{i2\pi/\ln(\lambda_1)}$. Consequently, we have the Fourier expansion

$$A_i(\beta_c-\beta)=\sum_{l\in\mathbf{Z}} a_{il}(\beta_c-\beta)^{i2\pi l/\ln(\lambda_1)} \tag{3.5}$$

At this point we have no additional information about these Fourier coefficients and possible restrictive relations among them. In the solvable examples[15] where the second possibility is realized the Fourier coefficients decrease exponentially with the mode number,[16] namely $|a_{il}|\propto e^{-u\omega|l|}$, where

$$\omega=2\pi/\ln(\lambda_1) \tag{3.6}$$

and u is a positive constant expected to be of order 1, but usually difficult to calculate. If a similar suppression occurs in the problem considered here, a truncation of the sum over the Fourier mode should provide acceptable approximations (see Section 5 for an example).

If we consider the new parametrization of the susceptibility, with the constants replaced by sums over Fourier modes, we obtain a parametrization of the HT coefficients as a linear combination of terms of the form $(\beta_c-\beta)^z$. The asymptotic (at large m) form of the coefficients is obtained from

$$(\beta_c-\beta)^z=\beta_c^z\sum_{m=0}^{\infty}\binom{z}{m}(-1)^m\left(\frac{\beta}{\beta_c}\right)^m \tag{3.7}$$

and the asymptotic form

$$\binom{z}{m}(-1)^m=\frac{m^{-z-1}}{\Gamma(-z)}\left(1+\frac{z+z^2}{2m}+\frac{2z+9z^2+10z^3+3z^4}{24m^2}\right.$$
$$\left.+\frac{6z^2+17z^3+17z^4+7z^5+z^6}{48m^3}+\cdots\right) \tag{3.8}$$

From this we obtain the following asymptotic form for the coefficients:

$$\begin{aligned}b_m = {} & m^{\gamma-1} \sum_{l\in \mathbf{Z}} K_l m^{il\omega}\{1+[(\gamma+il\omega)^2-(\gamma+ilw)]/2m+\cdots\}\\ & + m^{\gamma-\Delta-1}\sum_{l\in \mathbf{Z}} L_l m^{il\omega}\{1+[(\gamma-\Delta+il\omega)^2\\ & -(\gamma-\Delta+il\omega)]/2m+\cdots\}+\cdots \end{aligned} \tag{3.9}$$

where the K_l and L_l are (unknown) coefficients proportional to the (unknown) Fourier coefficients. In the following we consider the case of truncated Fourier series where only K_0, $K_{\pm 1}$, and L_0 are nonzero. Plugging (3.9) into (3.1), expanding up to first order in K_1/K_0, L_0, and $1/m$, and neglecting terms of order L_0/m, we obtain

$$\begin{aligned}\hat{S}_m = {} & \gamma-1+2\,\mathrm{Re}[i(\omega+\omega^3)\, m^{i\omega}K_1/K_0]\\ & + L_0 m^{-\Delta}\{(\Delta^3-\Delta)+2\,\mathrm{Re}[(\Delta-\Delta^3-i\omega\\ & +3i\Delta^2\omega+3\Delta\omega^2-i\omega^3)\, m^{i\omega}K_1/K_0]\}\\ & + m^{-1}\,\mathrm{Re}[(\omega^2+5i\omega^2-2i\gamma\omega^3+7\omega^4-2\gamma\omega^4-i\omega^5)\, m^{i\omega}K_1/K_0] \end{aligned} \tag{3.10}$$

From the solvable examples we expect that $|K_2/K_0|$ should be of the same order as $|K_1/K_0|^2$. The corrections of this order to $\hat{S}_m$ read

$$2\,\mathrm{Re}[(2i\omega+8i\omega^3)\, K_2/K_0+(4i\omega^3-i\omega)(K_1/K_0)^2] \tag{3.11}$$

These corrections can be important at moderate ω (see Section 5). Importantly, we see that the $1/m$ terms have reappeared. In the case where $\omega \gg 1$ we see that all the oscillating terms are approximately in phase and proportional to $\mathrm{Re}[im^{i\omega}K_1/K_0]$. In the large-$\omega$ limit the $1/m$ corrections are enhanced by a factor ω^2 compared to the leading oscillating term. This feature will play an important role in the discussion of Section 6.

Before discussing the fits of the numerical values of the extrapolated slope for the Ising and the Landau–Ginzburg cases we will first show that the errors made in the numerical calculations do not play any significant role and then discuss the fitting strategy with a solvable example.

4. THE EFFECT OF VOLUME AND ROUNDOFF ERRORS

In this section we discuss the errors made in the calculation of the coefficients and show that they have no relevant effect on the extrapolated slope for the discussion which follows. There are two sources of errors:

the numerical roundoffs and the finite number of sites. We claim that with 2^{100} sites and $D=3$ the finite-volume effects are several order of magnitude smaller than the roundoff errors.

From Eq. (2.2) one sees that the leading volume dependence will decay like $(c/2)^n$. This observation can be substantiated by using exact results at finite volume[17] for low-order coefficients or by displaying the values of higher order coefficients at successive iterations as in Fig. 1 of ref. 11. In both cases we observe that the $(c/2)^n$ law works remarkably well. For the calculations presented here we have used $c=2^{1/3}$ (i.e., $D=3$) and $n=100$, which gives volume effects on the order of 10^{-20}.

On the other hand, the roundoff errors are expected to grow like the square root of the number of arithmetical operations. In ref. 11 we estimated this number as approximately nm^2 for a calculation up to order m in the high-temperature expansion with 2^n sites. Assuming a typical roundoff error in double precision of the order of 10^{-17} and $n=100$, we estimate that the error on the mth coefficient will be of order $m\times 10^{-16}$ (or, more conservatively, bounded by $m\times 10^{-15}$). We have verified this approximate law by calculating the coefficients using a rescaled temperature and undoing this rescaling after the calculation. We chose the rescaling factor to be 0.8482. The rescaled critical temperature is then approximately 1. This prevents the appearance of small numbers in the calculation. If all the calculations could be performed exactly, we would obtain the same results as with the original method. However, for calculations with finite precision the two calculations have independent roundoff errors. The difference between the coefficients obtained with the two procedures is shown in Fig. 1 and is compatible with the approximate law. From this we conclude that for $m\leqslant 1000$ the errors on the coefficients should not exceed 10^{-12}.

We are now left with the task of estimating the effects that the errors on the b_m have on $\hat{S}_m$. In general, $\hat{S}_m$ is a function of b_m, b_{m-1}, b_{m-2}, and b_{m-3}. We will use the linear estimated error $\delta\hat{S}_m=\sum_{i=0,\ldots,3}(\partial\hat{S}_m/\partial b_{m-i})\,\delta b_{m-i}$. The derivatives of $\hat{S}_m$ with respect to these four variables are lengthy expressions which can be calculated from Eq. (3.1). The numerical values of these derivatives are shown in Fig. 2 for a Ising measure. Using an upper bound of 10^8 for the absolute value of the derivative and an upper bound of 10^{-12} on the absolute value of the errors on the coefficients, we find that none of the four terms of $\delta\hat{S}_m$ should exceed 10^{-4} in absolute value. Since the effects discussed later are typically on the order of 0.1, such errors will not play any role in the following.

We have found independent checks of our error estimates. First, the smoothness of the data for the $\hat{S}_m$ rules out numerical fluctuations which would be visible on graphs. The amount of data for the calculations with

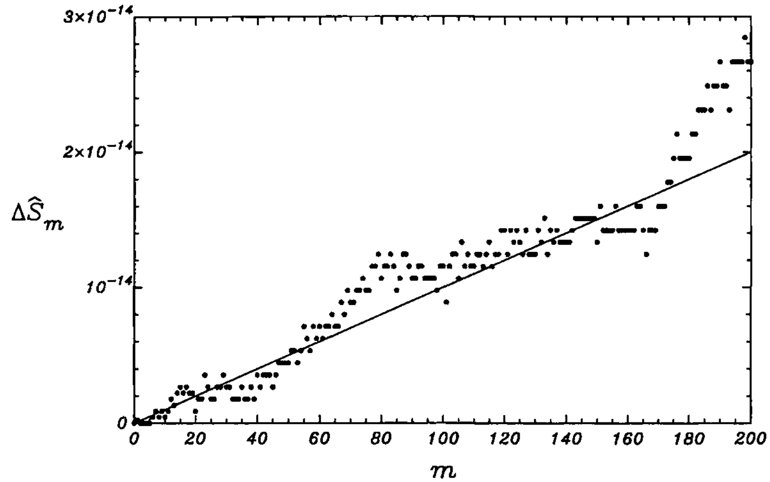

Fig. 1. Difference between the $\hat{S}_m$ calculated with the two procedures explained in the text. The solid line is $m \times 10^{-16}$.

a Landau–Ginzburg measure allows a visual resolutin of the order between 10^{-3} and 10^{-4}. Second, we have calculated $\hat{S}_m$ in the Gaussian case where nonzero results are of purely numerical origin. The results are displayed in Fig. 3. They show that the numerical fluctuations for the Gaussian hierarchical model are smaller than 10^{-7} for $m \leqslant 200$. This small number indicates that our previous estimates are conservative.

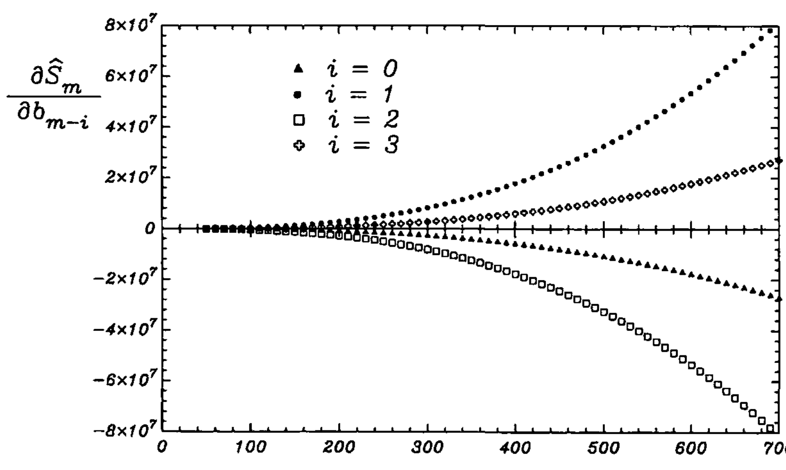

Fig. 2. Derivatives of $\hat{S}_m$ with respect to b_m, b_{m-1}, b_{m-2}, and b_{m-3} in the Ising case.

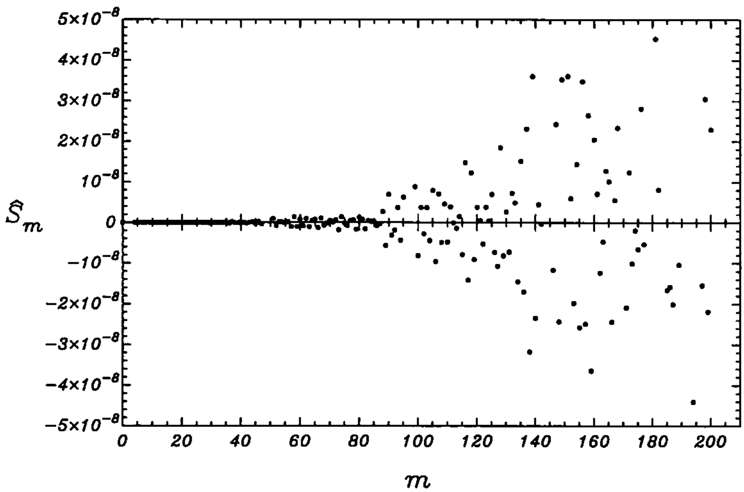

Fig. 3. $\hat{S}_m$ in the Gaussian case.

The calculation of the large m coefficients requires a lot of computing time. We found that using a truncation in the expansion in k at order 100 could cut the computer time by a factor of order 100 while having very small effects on the values of the coefficients. If we plot the differences between the values obtained with the truncated and the regular methods, we obtain a graph very similar to Fig. 3. For $m \leqslant 400$ the differences are less than 4×10^{-6}, which is compatible with the numerical errors discussed above. The data for the Landau–Ginzburg case presented here have been calculated with the truncated method.

5. DEVELOPING FITTING METHODS, WITH A SIMPLE EXAMPLE

The form of the coefficients given in Eq. (3.5) involves an infinite number of parameters. In order to see how one can obtain reasonable approximations with a manageable number of unknown parameters, we will first consider a simple example. One of the simplest examples of a function with a singularity and a log-periodic behavior is given by

$$G(x) = \sum_{n=0}^{\infty} \frac{B^n}{1 + A^n x} \tag{5.1}$$

This example has been motivated by the calculations of refs. 16 and the form of the analytic expressions corresponding to one-loop Feynman diagrams for the hierarchical model. For definiteness we shall only consider the case where A and B are real and $A > B > 1$.

Picking an arbitrary positive value x_0 and introducing a new variable $\beta \equiv 1 - x/x_0$, we obtain the "high-temperature expansion"

$$G(x) = \sum_{m=0}^{\infty} b_m \beta^m \tag{5.2}$$

with coefficients

$$b_m = \sum_{n=0}^{\infty} \frac{B^n A^{nm} x_0^m}{(1 + A^n x_0)^{m+1}} \tag{5.3}$$

The critical value of β is 1 and is obtained by setting $x = 0$ in its definition.

Using the Mellin transform technique discussed in refs. 16, we can rewrite

$$G(x) = G_{\text{reg}}(x) + G_{\text{sing}}(x) \tag{5.4}$$

with

$$G_{\text{reg}}(x) = \sum_{l=0}^{\infty} (-1)^l x^l (1 - BA^l)^{-1} \tag{5.5}$$

and

$$G_{\text{sing}}(x) = \frac{\omega}{2} x^{-a} \sum_{p=-\infty}^{+\infty} \frac{x^{-ip\omega}}{\sin(\pi(a + ip\omega))} \tag{5.6}$$

where we have used the notation

$$a = \frac{\ln B}{\ln A} \tag{5.7}$$

and

$$\omega = \frac{2\pi}{\ln A} \tag{5.8}$$

The complex part of the exponents comes from the fact that the Mellin transform of $G(x)$ has poles away from the real axis. Substituting $(\beta_c - \beta)\, x_0$ for x and considering a as a critical exponent, the analogy with the original problem becomes clear. Neglecting the regular part in (5.4) and proceeding as in Section 3, we obtain the asymptotic form of the coefficients as in (3.9), with γ replaced by a, $L_l = 0$, and

$$K_l = \frac{x_0^{-a}\pi}{\Gamma(a + i\omega l)\sin(\pi(a + i\omega l))} \tag{5.9}$$

For large $|l|$, the magnitude of the coefficients decreases like $[\exp(-\frac{1}{2}\pi\omega\,|l|)]\,|l|^{1/2-a}$. One sees that fast oscillations have small amplitudes and vice versa. This makes the oscillations hard to observe. In order to get an idea of how to obtain suitable truncations of the expansion given in Eq. (3.5), we have selected the values $A = 3$, $B = 10$, and $x_0 = 1$ and calculated the coefficients with the exact formula (5.3). The sums were truncated in such a way that the remainder would be less than 10^{-16}. We then started fitting the corresponding $\hat{S}_m$ using Eq. (3.1). We first used a truncation where the Fourier modes with $|l| \geqslant 2$ and corrections of order $1/m^2$ were dropped. We treated a, ω, and the complex number K_1/K_0 as unknown coefficients and determined their values by minimizing the sum of the square of the errors with Powell's method. This allowed us to determine the order of magnitude of ω and a. Plotting the difference between the best fit and the exact values versus the logarithm of m shows oscillations twice as rapid as the oscillations in the fit. In other words, we needed the $l = \pm 2$ terms. With these terms included and using the data for $m \geqslant 30$ we obtained $\omega = 2.727$ and $a = 0.4772$, in agreement with the exact values given by Eqs. (5.7) and (5.8), with three significant digits. The data and the fit are shown in Fig. 4. In this simple example we found that each

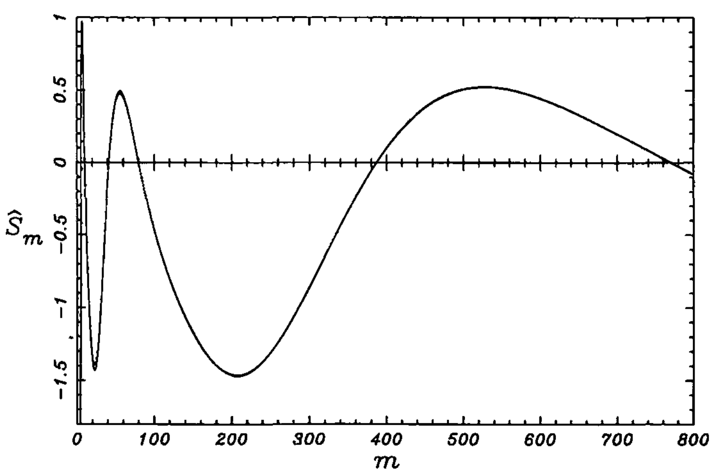

Fig. 4. $\hat{S}_m$ for the example of Section 5 with $A = 3$, $B = 10$, and $x_0 = 1$ and the fit described in the text.

correction taken into account improved the quality of the fits. This is related to the fact that ω takes a not too large value. As we now proceed to discuss, a substantially larger value of ω implies a rather more complicated situation.

6. FITTING THE EXTRAPOLATED SLOPE

We now discuss the fits of the extrapolated slope for Dyson's hierarchical model. The data are shown in Fig. 5 for the various measures considered. From the equally spaced oscillations in the $\ln(m)$ variable one finds immediately that ω is approximately 18. According to the exponential suppression hypothesis, this large value makes plausible that only the Fourier modes with $|l| \leqslant 1$ should be kept. This simplification unfortunately has the counterpart that for large ω the $1/m$ expansion is effectively an ω^2/m expansion, as explained at the end of Section 3.

To be more specific, the relative strength of the leading oscillations and their $1/m$ corrections is approximately $\omega^2/(2m)$. For $\omega = 18$ this means that for $m = 162$ the leading term and the first corrections have the same weight. In the example considered in the previous section the critical value was $m = 4$ and good-quality fits in the asymptotic region required considering the data for values of m larger than about ten times this critical value—which represents dropping only 5% of the data. For the hierarchical model our data are limited to five times the critical value. Consequently, we

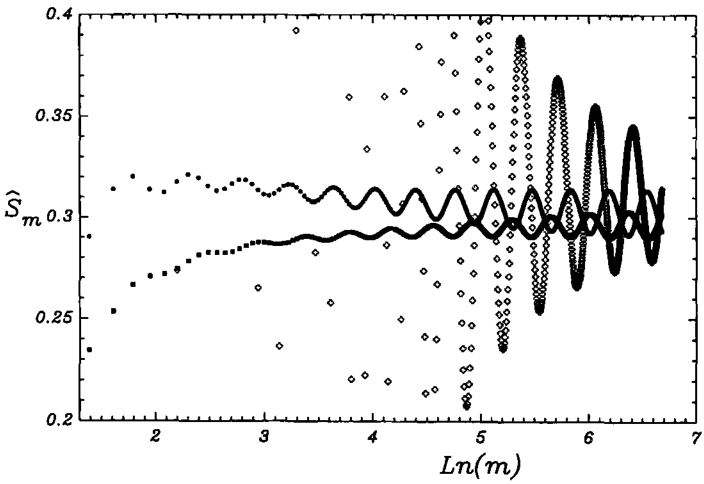

Fig. 5. $\hat{S}_m$ for the Ising model (crosses) and the Landau–Ginzburg model with $B = 1$ (circles) and $B = 0.1$ (squares).

probably need about 2500 coefficients in order to get results as accurate as in the example of Section 5. Despite the fact that we do not expect Eq. (3.10) to be very accurate for the existing data, we will use an unbiased parametrization of the extrapolated slope which can be recast in the form (3.10) when some special constraints on the parameters are imposed. An unbiased parametrization of the form

$$\hat{S}_m = \gamma - 1 + a_1 m^{-a_2} \cos(\omega \ln(m) + a_3) + a_4 m^{-a_2} + a_5 \cos(\omega \ln(m) + a_6) \\ + a_7 m^{-1} \cos(\omega \ln(m) + a_8) \qquad (6.1)$$

gives very good quality fits provided that we disregard the low-m data (see below). An example of such a fit is displayed in Fig. 6. The difference between the data and the fit is barely visible for $m \geqslant 100$. For $m \leqslant 100$, where we do not have any reason to believe in the validity of the $1/m$ expansion, the frequency is still well fitted, but not the amplitude. The assumption that only the Fourier modes with $|l| \leqslant 1$ should be retained can be checked explicitly from the fact that the differences between the fit and the data do not show more rapid oscillations (unlike in the previous section, where the $|l| = 2$ modes were important).

We have tried to use the fits based on Eq. (6.1) to test the validity of the more restricted parametrization (3.10) obtained from the $1/m$ expansion. To be more specific, Eqs. (3.10) and (6.1) have two common parameters (γ and ω), while the eight a_p parameters of (6.1) are expressed in terms of one complex (K_1/K_0) and two real (Δ and L_0) parameters in

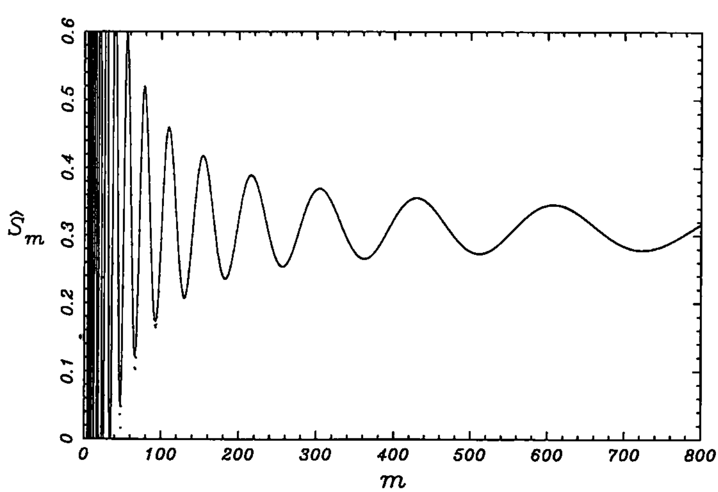

Fig. 6. $\hat{S}_m$ for the Ising model and a ten-parameter fit.

Eq. (3.10). Consequently, Eq. (3.10) imposes four independent relations among the a_p. In the limit of large ω these relations are relatively simple: $a_3 = a_6 = a_8$, $2a_7 = -\omega^2 a_5$, and $a_4 a_5 = a_1 a_2(1 - a_2^2)$. We used the data for $m > m_{\min}$, with $m_{\min}$ larger than 200, and varied $m_{\min}$ and the initial values of the parameters. We found that for the fits based on Eq. (6.1), or their restriction to the case where all the phases of the oscillatory terms are taken equal, the values of the fitted parameters depend sensitively on the value of $m_{\min}$ and on the initial values. Sampling some of the many solutions, we found no indications that all the relations dictated by Eq. (3.10) were approximately obeyed.

We have nevertheless been able to design a stable procedure with fewer parameters. To assess the stability, we vary $m_{\min}$ between 200 and 400, keeping $m_{\max}$ at 800. The upper value of $m_{\min}$ is chosen in such a way that we have at least two complete oscillations. We first set a_4, a_5, and a_7 equal to zero, which yields a parametrization of $\hat{S}_m$ as in Eq. (3.4). These restricted fits do not suffer from the sensitive dependence mentioned above. We then analyze the errors as a function of m. In all the cases considered the difference between the fit and the data is much smaller than the amplitude of the oscillations (for $m \geqslant 200$) and can be approximated by a constant plus a negative power of m. Putting together the fit of the extrapolated slope and the fit of the differences, we were able to obtain six-parameter fits with a good stability and small systematic errors in γ. We now discuss the two cases separately.

In the Ising case the decay of the oscillations controlled by m^{-a_2} in the five-parameter fit and the decay of the errors are both approximately $m^{-0.6}$. We thus decided to use Eq. (6.1) with a_5 and a_7 equal to zero (making a_6 and a_8 irrelevant). The six-parameter fits so obtained are then reasonably stable under small changes in $m_{\min}$ (see Figs. 8 and 9). Nevertheless, a systematic tendency can be observed: when $m_{\min}$ is varied between 200 and 400, a_2 evolves slowly from 0.67 to 0.57. It is conceivable that if we had data at larger m, a_2 would evolve toward its expected value 0.46.

In the Landau–Ginzburg case the value of a_2 obtained from the five-parameter fits is very small and the amplitude is in first approximation constant. We thus set a_1 and a_7 equal to zero, while a_5 parametrizes the amplitude of the oscillations and a_4 corrects the systematic errors. The power a_2 does not have the smooth behavior under a change of $m_{\min}$ it had in the Ising case; however, it does the job that it is required to do: the errors are small and do not show any kind of tilt or period doubling. These errors are displayed on Fig. 7. Their order of magnitude is 10^{-3}, which can be used as a rough estimate of our systematic errors. Statistical errors due to the roundoff errors are visible on the right part of the graph and are clearly smaller by at least one order of magnitude. We now proceed to

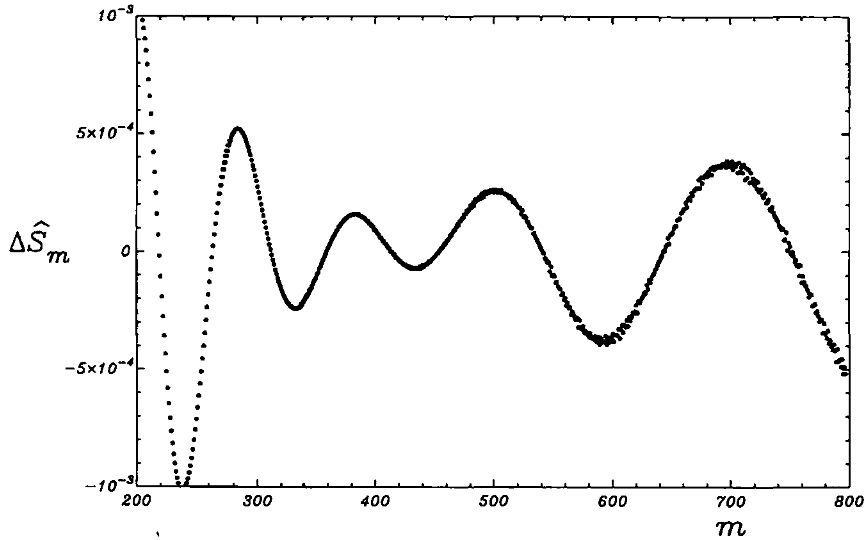

Fig. 7. Difference between $\hat{S}_m$ for Landau-Ginzburg with $B=1$ and the fit given by Eq. (6.1) with $\gamma=1.30137$, $\omega=17.716$, $a_1=a_7=0$, $a_5=-0.01084$, $a_6=0.3367$, $a_4=0.917$, and $a_2=1.0589$.

discuss the estimation of the most important quantities (γ and ω) from these fits.

7. ESTIMATION OF γ AND ω AND COMPARISON WITH EXISTING RESULTS

The values of γ as a function of $m_{\min}$ are displayed in Fig. 8. The mean values are 1.3023 in the Ising case and 1.2998 (1.2978) in the Landau-Ginzburg case with $B=1$ ($B=0.1$). We conclude that $\gamma=1.300$ with a systematic error of the order of 0.002. As explained in the previous section, a precise estimation of the subleading exponents seems difficult.

Our results can be compared with those obtained from the ε-expansion,[10] namely $\lambda_1=1.427$ and $\lambda_2=0.85$. These results imply $\gamma=1.300$ and $\Delta=0.46$. We have checked these results with methods which do not rely on the ε-expansion or expansions in the renormalized coupling constants. First, we have adapted a numerical method discussed in refs. 1 and 2 to the case of the hierarchical model. We obtained $\lambda_1=1.426$. Second, we have used a truncated and rescaled[11] version of Eq. (2.2) which corresponds to the usual renormalization group transformation. Using fixed values of beta and retaining only terms of order up to k^{28} at

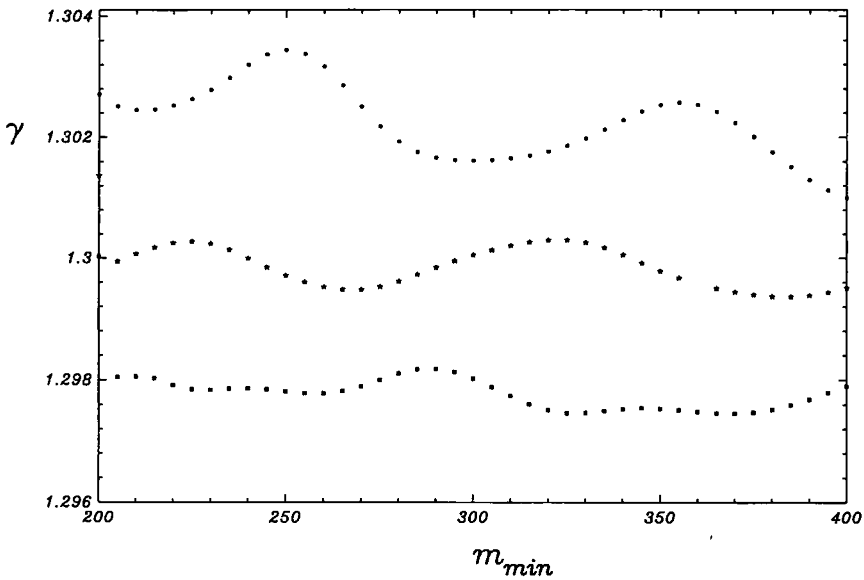

Fig. 8. γ as a function of $m_{\min}$, with $m_{\min}$ between 200 and 400 by steps of 5 for the Ising model (circles) and the Landau–Ginzburg model with $B=1$ (stars) and $B=0.1$ (squares).

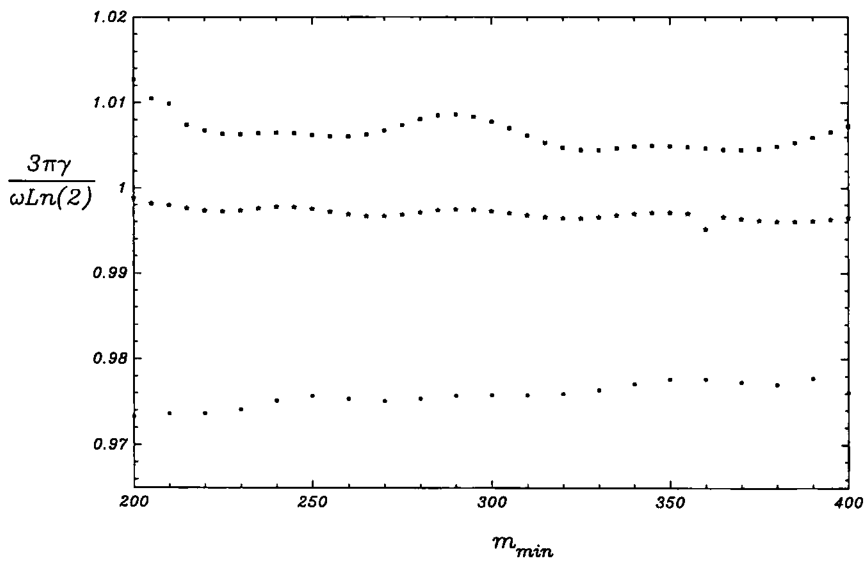

Fig. 9. Plot of $3\pi\gamma/[\omega \ln(2)]$ as a function of $m_{\min}$ (as in Fig. 8) for the Ising model (circles) and the Landau–Ginzburg model with $B=1$ (stars) and $B=0.1$ (squares).

each step of the calculation, we were able to determine the fixed point and the linearized renormalization group transformation in this 14-dimensional subspace. Diagonalizing this matrix, we found $\lambda_1 = 1.426$ and $\lambda_2 = 0.853$. The corresponding value of γ is 1.302. For both methods the errors can be estimated by comparing the linearizations obtained for successive iterations near the fixed point. The order of magnitude of these errors is 0.001 in both cases. As a by-product we also found that all the other eigenvalues were (robustly) real, ruling out the possibility of complex eigenvalues.

We now consider the values of ω. A distinct signature of the "second possibility" discussed in refs. 1, 13, 15, and 16 is the relation

$$\omega = \frac{3\pi}{\ln 2}\gamma \tag{7.1}$$

This relation is well-obeyed by the *a priori* independent quantities used in the fits, as shown in Fig. 9.

8. OPEN QUESTIONS AND CONCLUSIONS

We have thus succeeded in finding a theoretical framework in which the new and existing results appear compatible within errors of the order of 0.002. In addition, we also have a qualitative understanding of the behavior of the extrapolated slope in the low-m region. Many questions remain to be answered. First, we would like to understand the origin of the oscillations. If the example of the solvable Ising hierarchical lattice models can be used as a guide, the oscillations are due to poles of the Mellin transform located away from the real axis. These poles are related to an accumulation of singularities at the critical point. We have tried to get an indication that a similar mechanism would be present for the models considered here. As a first step we have calculated the expansion of the partition function about $\beta = 1.179$, a good estimate[18] of the critical temperature. We have carried the expansion up to order 10 for 2^n sites with $n = 6$–12. The zeros are displayed in Fig. 10. It appears that the approximate half-circle on which they lie shrinks around the critical point when the volume increases. It is not clear that the polynomial expansion is a good approximation. This could in principle be checked by searching for the exact zeros. However, this is a much harder calculation because, due to the existence of couplings of different strengths, the partition function cannot be written as a polynomial in a single variable of the form $e^{v\beta}$.

The existence of log-periodic corrections to a singular behavior seems to be a feature of hierarchically organized systems. Empirical observations

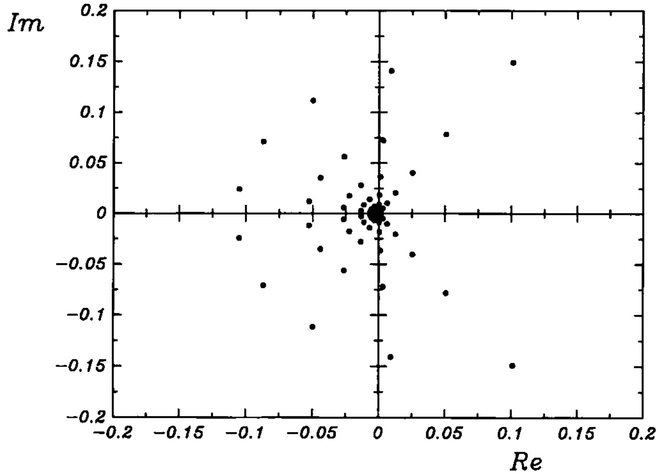

Fig. 10. The zeros of the partition function in the complex temperature plane, in the Ising case with from 2^6 to 2^{12} sites. The origin on the graph represents the point $\beta = 1.179$. The outer set of points (on an approximate ellipse) is for $n = 6$, the next set for $n = 7$, etc.

of such a phenomena have been suggested as a possible way to predict the occurrence of earthquakes[19] and stock market crashes.[20] Are similar phenomena present for translationally invariant models with nearest neighbor interactions? Using the longest series available[6] for a nearest neighbor model, namely the two-dimensional Ising model on a square lattice, we found no clear evidence for regular log-periodic oscillations comparable to those seen in Fig. 6. However, the situation is complicated by the existence an antiferromagnetic point at $\beta = -\beta_c$. We used an Euler transformation as discussed in ref. 6 to eliminate this problem and found no indications of oscillations having a period that increases with m. On the other hand, the zeros of the partition function in the complex temperature plane have been studied[21] extensively. The zeros appear on two circles in the $\tanh(\beta)$ plane, one of them going through the ferromagnetic critical point. Thus it seems incorrect to conclude that any accumulation of singularities will create oscillations. Approximate calculations of the Mellin transform of the susceptibility of the two-dimensional Ising model could shed some light on this question.

We also would like to be able to calculate the amplitudes of oscillation with a method independent of the high-temperature expansion. As explained in the introduction, the study of the linearized renormalization group transformation does not provide any indications concerning the oscillations. Up to now the *global* properties of the flows are only accessible through numerical approaches. The results presented here should be seen as an encouragement to develop and test global approaches to the renormalization group flows.

ACKNOWLEDGMENTS

We thank V. G. J. Rodgers for his help regarding computational problems. One of us (Y.M.) stayed at the Aspen Center for Physics during the last stage of this work and benefited from illuminating conversations with Martin Block concerning the fits of Section 6.

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